# Hom-Gerstenhaber algebras and Hom-Lie algebroids

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Satyendra Kumar Mishra
Statistics & Mathematics Unit,
ISI Bangalore.

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# Introduction

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- The first appearance of such an algebra was the notion of hom-Lie algebra in the context of $q$-deformations of Witt and Virasoro algebras.
- In this talk, we are interested in a geometric generalisation of hom-Lie algebras, the notion of hom-Lie algebroid.

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- In recent years, there is a growing interest in new non-associative algebra structures, known as hom-algebraic structures due to their close relationship with deformed vector fields and differential calculus.
- The first appearance of such an algebra was the notion of hom-Lie algebra in the context of $q$-deformations of Witt and Virasoro algebras.
- In this talk, we are interested in a geometric generalisation of hom-Lie algebras, the notion of hom-Lie algebroid.
- Hom-Lie algebroids were defined through a formulation of hom-Gerstenhaber algebras.

# Content List

This talk is divided into three parts:

- Gerstenhaber algebras & Lie algebroids;
- Hom-Lie algebra structures;
- Relationship between hom-Gerstenhaber algebras & hom-Lie algebroids.

# I. Gerstenhaber Algebras & Lie Algebroids

# Gerstenhaber algebras

Definition: [M. Gerstenhaber, 1964]

A Gerstenhaber algebra is a triplet $(\mathcal{A} = \bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i, \wedge, [-,-])$, where $\mathcal{A}$ is a graded commutative associative $\mathbb{K}$-algebra and $[-,-] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a bilinear map of degree $-1$ such that

- $(\mathcal{A}[1], [-,-])$ is a graded Lie algebra.
  (Here $\mathcal{A}[1]_i = \mathcal{A}_{i+1}$ for all $i \in \mathbb{Z}$).
- The following Leibniz rule holds:

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{(i-1)j} Y \wedge [X, Z],$$

for all $X \in \mathcal{A}_i,\ Y \in \mathcal{A}_j,\ Z \in \mathcal{A}_k$.

# Gerstenhaber algebras

- A Gerstenhaber algebra $(\mathcal{A}, \wedge, [-,-])$ is said to be an exact Gerstenhaber algebra if there exists a square zero operator $D : \mathcal{A} \to \mathcal{A}$ of degree $-1$ such that $[X, Y] = (-1)^{|X|}\big(D(X \wedge Y) - D(X) \wedge Y - (-1)^{|X|} X \wedge D(Y)\big)$ for any homogeneous elements $X, Y \in \mathcal{A}$.

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- A differential Gerstenhaber algebra is a Gerstenhaber algebra $(\mathcal{A}, \wedge, [-,-])$ equipped with a square zero map $d : \mathcal{A} \to \mathcal{A}$ such that $d$ is a derivation of degree 1 with respect to the product $\wedge$. Moreover, if $d$ is a derivation of the graded Lie bracket then the Gerstenhaber algebra is said to be "strong differential Gerstenhaber algebra".

# Lie algebroids

Definition: [Mackenzie, 2005]

A Lie algebroid over a smooth manifold $M$ is a vector bundle $A$ over $M$ equipped with a bundle map $\rho : A \to TM$, called the anchor map, and a bilinear map $[-,-] : \Gamma A \otimes \Gamma A \to \Gamma A$ such that

- $(\Gamma A, [-,-])$ is a Lie algebra;
- $[X, f.Y] = f.[X, Y] + \rho(X)(f).Y$ for all $X, Y \in \Gamma A$ and $f \in C^\infty(M)$.

If $A$ is a rank $n$ vector bundle over a smooth manifold $M$ and $\oplus_{0 \leq k \leq n} \Gamma(\wedge^k A)$ is the exterior algebra of multisections of the bundle $A$. Then,

# Gerstenhaber algebras and Lie algebroids

| Geometric structures on rank $n$ vector bundle $A$ | Gerstenhaber algebra structures on the space $\oplus_{k\geq 0}\Gamma(\wedge^k A)$ |
|---|---|
| Lie algebroids | Gerstenhaber algebras |
| Lie algebroids with a flat connection on the line bundle $\wedge^n A$ | Exact Gerstenhaber algebras (BV algebras) |
| Lie bialgebroids | Strong differential Gerstenhaber algebras |

## II. Hom-Lie algebra structures

# Hom-Lie algebras

Definition: [J. Hartwig, D. Larsson, and S. Silvestrov, 2006]

A hom-Lie algebra is a triplet $(\mathfrak{g}, [-,-], \alpha)$ where $\mathfrak{g}$ is a $\mathbb{K}$-vector space equipped with

- a skew-symmetric $\mathbb{K}$-bilinear map $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ and
- a linear map $\alpha : \mathfrak{g} \to \mathfrak{g}$

such that

- $\alpha[x, y] = [\alpha(x), \alpha(y)]$ and
- $[\alpha(x), [y, z]] + [\alpha(y), [z, x]] + [\alpha(z), [x, y]] = 0$

for all $x, y, z \in \mathfrak{g}$.

## Example

Let $(\mathfrak{g}, [-,-])$ be a Lie algebra, and $\alpha : \mathfrak{g} \to \mathfrak{g}$ be a Lie algebra homomorphism, then the triplet $(\mathfrak{g}, [-,-]_\alpha, \alpha)$ is a hom-Lie algebra, where

$$[x, y]_\alpha := [\alpha(x), \alpha(y)]$$

for any $x, y \in \mathfrak{g}$.

## $q$-Deformations: Pioneering works


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- Deformations (Makhlouf, Silvestrov, Ammar, Ejbehi)
- Integration of hom-Lie algebras (Laurent-Gengoux, Makhlouf, and Teles)
- Hom-Gerstenhaber algebras and Hom-Lie algebroids (Laurent-Gengoux, and Teles).

# Hom-Gerstenhaber algebras

Definition: [Laurent-Gengoux and Teles, 2013]

A hom-Gerstenhaber algebra is a quadruple $(\mathcal{A}, \wedge, [-,-], \alpha)$, where $\mathcal{A} = \oplus_{i\in\mathbb{Z}}\mathcal{A}_i$ is a graded commutative associative $\mathbb{K}$-algebra, map $\alpha$ is an endomorphism of $(A, \wedge)$ of degree 0, and $[-,-] : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ is a bilinear map of degree $-1$ such that

- $(\mathcal{A}[1] = \oplus_{i\in\mathbb{Z}}\mathcal{A}_{(i+1)}, [-,-], \alpha)$ is a graded hom-Lie algebra.
- The hom-Leibniz rule holds:

$$[X, Y \wedge Z] = [X, Y] \wedge \alpha(Z) + (-1)^{(i-1)j}\alpha(Y) \wedge [X, Z]$$

for all $X \in \mathcal{A}_i, \; Y \in \mathcal{A}_j, \; Z \in \mathcal{A}_k$.

## Example-I

Let $(\mathcal{A}, [-,-], \wedge)$ is a Gerstenhaber algebra and a map

$$\alpha : (\mathcal{A}, [-,-], \wedge) \to (\mathcal{A}, [-,-], \wedge)$$

be an endomorphism of Gerstenhaber algebras, then the quadruple $(\mathcal{A}, \wedge, [-,-]_\alpha, \alpha)$ is a hom-Gerstenhaber algebra, where the graded hom-Lie bracket $[-,-]_\alpha$ is given as follows

$$[X, Y]_\alpha = [\alpha(X), \alpha(Y)] \quad \text{for all } X, Y \in \mathcal{A}.$$

## Example II

Let $(\mathfrak{g}, [-,-], \alpha)$ be a hom-Lie algebra, then one obtains a canonical hom-Gerstenhaber algebra $(\mathfrak{G} = \wedge^* \mathfrak{g}, \wedge, [-,-]_{\mathfrak{G}}, \alpha_{\mathfrak{G}})$, where

$$[x_1 \wedge \cdots \wedge x_n, y_1 \wedge \cdots \wedge y_m]_{\mathfrak{G}}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (-1)^{i+j} [x_i, y_j] \wedge \alpha_G (x_1 \wedge \cdot\cdot \hat{x}_i \cdot\cdot \wedge x_n \wedge y_1 \wedge \cdot\cdot \hat{y}_j \cdot\cdot \wedge y_m)$$

for any $x_1, \cdots, x_n, y_1, \cdots, y_m \in \mathfrak{g}$, and the map $\alpha_G : \mathfrak{G} \to \mathfrak{G}$ is given as follows:

$$\alpha_{\mathfrak{G}}(x_1 \wedge \cdots \wedge x_n) = \alpha(x_1) \wedge \cdots \wedge \alpha(x_n).$$

# Hom-bundle

Definition: [L. Cai, J. Liu, and Y. Sheng, 2017]

A hom-bundle is a triplet $(A, \psi, \alpha)$ equipped with a vector bundle $A$ over $M$, a smooth map $\psi : M \to M$, and a linear map $\alpha : \Gamma A \to \Gamma A$ such that the following condition is satisfied.

$$\alpha(f.x) = \psi^*(f).\alpha(x)$$

for all $f \in C^\infty(M)$ and $x \in \Gamma A$. We say that $\alpha : \Gamma A \to \Gamma A$ is a $\psi^*$-function linear map.

# Hom-Lie algebroids

Definition: [Laurent-Gengoux and Teles, 2013]

A hom-Lie algebroid over $M$ is a hom-bundle $(A, \psi, \alpha)$ equipped with a bilinear map $[-,-] : \Gamma A \otimes \Gamma A \to \Gamma A$ and a vector bundle morphism $\rho : \psi^! A \to \psi^! TM$ such that following conditions hold.

- The triplet $(\Gamma A, [-,-], \alpha)$ is a hom-Lie algebra.
- The pair $(\rho, \psi^*)$ is a representation of $(\Gamma A, [-,-], \alpha)$ on $C^\infty(M)$.
- For all $s, t \in \Gamma A,\ f \in C^\infty(M)$,

$$[s, f.t] = \psi^*(f).[s, t] + \rho(s)[f].\alpha(t).$$

# III. Relationship Between
# Hom-Gerstenhaber Algebras & Hom-Lie Algebroids

# A Bijective Correspondence

Theorem: [Laurent-Gengoux and Teles, 2013]

There is a bijective correspondence between hom-Lie algebroid structures on the hom-bundle $(A, \psi, \alpha)$ and hom-Gerstenhaber algebra structures on the pair $(\mathfrak{A}, \tilde{\alpha})$, where

$$\mathfrak{A} = \bigoplus_{k \geq 0} \Gamma(\wedge^k A)$$

and the map $\tilde{\alpha}$ is an extension of $\alpha$ to higher degree elements.

## Hom-Batalin Vilkovisky algebras

Definition: [A. Mandal and S.K. Mishra, 2018]

A hom-Gerstenhaber algebra $\mathcal{A} = (\bigoplus_{i\in\mathbb{Z}} \mathcal{A}_i, \wedge, [-,-], \alpha)$ is said to be generated by an operator $D : \mathcal{A} \to \mathcal{A}$ of degree $-1$ if $D \circ \alpha = \alpha \circ D$, and

$$[X, Y] = (-1)^{|X|}(D(XY) - (DX)\alpha(Y) - (-1)^{|X|}\alpha(X)(DY))$$

for homogeneous elements $X, Y \in \mathcal{A}$. Furthermore, if $D^2 = 0$, then $D$ is called an exact generator and the hom-Gerstenhaber algebra $\mathcal{A}$ is called an exact hom-Gerstenhaber algebra or hom-Batalin-Vilkovisky algebra.

# Example

Let $(\mathfrak{G} = \wedge^* \mathfrak{g}, \wedge, [-,-]_{\mathfrak{G}}, \alpha_{\mathfrak{G}})$ be the hom-Gerstenhaber algebra associated to a hom-Lie algebra $(\mathfrak{g}, [-,-], \alpha)$. Let us consider a boundary operator $d : \wedge^n \mathfrak{g} \to \wedge^{n-1} \mathfrak{g}$ of a hom-Lie algebra with coefficients in the trivial module $\mathbb{K}$:

$$d(x_1 \wedge \cdots \wedge x_n) = \sum_{1 \leq i < j \leq n} (-1)^{i+j} [x_i, x_j] \wedge \alpha_G (x_1 \wedge \cdot\cdot \hat{x}_i \wedge \cdot\cdot \wedge \hat{x}_j \wedge \cdot\cdot \wedge x_n)$$

for all $x_1, \cdots, x_n \in \mathfrak{g}$ and $n > 1$. Then, $d : \mathfrak{G} \to \mathfrak{G}$ is a map of degree $-1$ such that $d^2 = 0$, and $d \circ \alpha = \alpha \circ d$. This operator $d$ generates the bracket $[-,-]_{\mathfrak{G}}$, i.e.

$$[X, Y]_{\mathfrak{G}} = (-1)^{|X|} (d(XY) - (dX) \alpha_{\mathfrak{G}}(Y) - (-1)^{|X|} \alpha_{\mathfrak{G}}(X)(dY))$$

## Strong Differential hom-Gerstenhaber algebras

Definition: [Mandal, A, and Mishra, SK, 2018]

A hom-Gerstenhaber algebra $\mathfrak{A} := (\bigoplus_{i\in\mathbb{Z}}\mathcal{A}_i, \wedge, [-,-], \alpha)$ is called a differential hom-Gerstenhaber algebra if it is equipped with a degree 1 map $d : \mathfrak{A} \to \mathfrak{A}$ such that

- $d$ is an $(\alpha, \alpha)$-derivation of degree 1 with respect to $\wedge$, i.e.
$$d(X \wedge Y) = d(X) \wedge \alpha(Y) + (-1)^{|X|}\alpha(X) \wedge d(Y)$$
for any $X, Y \in \mathfrak{A}$.

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for any $X, Y \in \mathfrak{A}$.
- $d^2 = 0$, and the map $d$ commutes with $\alpha$.

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- $d$ is an $(\alpha, \alpha)$-derivation of degree 1 with respect to $\wedge$, i.e.
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for any $X, Y \in \mathfrak{A}$.
- $d^2 = 0$, and the map $d$ commutes with $\alpha$.

Hom-Gerstenhaber algebra $\mathfrak{A}$ is said to be a strong differential hom-Gerstenhaber algebra if $d$ also satisfies the equation:

$$d[X, Y] = [dX, \alpha(Y)] + [\alpha(X), dY] \text{ for any } X, Y \in \mathfrak{A}.$$

# Bijective correspondences

| Geometric structures on rank $n$ hom-bundle $(A, \psi, \alpha)$ | Hom-algebraic structures on the pair $(\mathfrak{A}, \tilde{\alpha})$ |
|---|---|
| Hom-Lie algebroids | Hom-Gerstenhaber algebras |
| * Hom-Lie algebroids with a representation on the associated hom-bundle $(\wedge^n A, \psi, \tilde{\alpha})$ | * Exact hom-Gerstenhaber algebras (Hom-BV algebras) |
| **Hom-Lie bialgebroids | **Strong differential hom-Gerstenhaber algebras |

Note: * and ** requires invertibility of the hom-bundle.

# Bijective correspondences

Theorem: [A. Mandal and S.K. Mishra, 2018]

If $(A, \psi, \alpha)$ be an invertible rank $n$ hom-bundle, then we have the following bijective correspondences:

- Hom-BV algebra structures on $(\mathfrak{A}, \tilde{\alpha}) \leftrightarrow$ hom-Lie algebroid structures on the hom-bundle $(A, \psi, \alpha)$ with a representation of this hom-Lie algebroid on the hom-bundle $(\wedge^n A, \psi, \tilde{\alpha})$.

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- Hom-BV algebra structures on $(\mathfrak{A}, \tilde{\alpha}) \leftrightarrow$ hom-Lie algebroid structures on the hom-bundle $(A, \psi, \alpha)$ with a representation of this hom-Lie algebroid on the hom-bundle $(\wedge^n A, \psi, \tilde{\alpha})$.
- Strong differential hom-Gerstenhaber algebra structures on $(\mathfrak{A}, \tilde{\alpha})$ $\leftrightarrow$ hom-Lie bialgebroid structures on the hom-bundle $(A, \psi, \alpha)$.

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Thank You!